Math 275D Lecture 8 Notes

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1 Tail Events, Limsups, and Stopping Times for Brownian Motion

1.1 Tail events do not depend on starting point

If $A \in \mathcal{T}$, we know that $\mathbb{P}_x(A) = 0$ or 1. On the other hand, this may depend on x (i.e. it is a function g of x). We want to show that for Brownian motion, $g(x) = g(0)$ for all x.

Example 1.1. Consider the event

$$A = \left\{ \limsup_{n \text{ is prime}} \frac{B(n)}{\sqrt{n}} \leq \frac{1}{2} \right\}.$$

Since A is in the tail σ-field $\mathcal{T}$, $\mathbb{1}_A = \mathbb{1}_D \circ \theta_1$ for a shifted event D. Then

$$\begin{aligned}
\mathbb{P}_0(A) &= \mathbb{E}[\mathbb{1}_A] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_1]] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{1}_A \circ \theta_1 \mid \mathcal{F}_1]] \\
&= \mathbb{E}[\underbrace{\mathbb{E}_{B(1)}[\mathbb{1}_D]}_{:=\varphi(B(1))}] \\
&= \int \varphi(a) p_1(0,a)\, da.
\end{aligned}$$

We know that $0 \leq \varphi(B(1)) \leq 1$. Since $p_1(0,a)$ is positive and we know that $\mathbb{P}_0(A) = 0$ or 1, we must have $\varphi(a) = \mathbb{P}_0(A)$ for a.e. a.

The same argument starting at x gives

$$\mathbb{P}_x(A) = \int \varphi(a) p_1(x,a)\, da = \mathbb{P}_0(A) \int p_1(x,a)\, da = \mathbb{P}_0(A).$$

1.2 Limsup of Brownian motion

Let's try to show that

$$\limsup_t \frac{B_t}{\sqrt{t}} = \infty \quad \text{a.s.}$$

By symmetry, this will also mean that

$$\liminf_t \frac{B_t}{\sqrt{t}} = -\infty \quad \text{a.s.}$$

Let

$$f(k) = \mathbb{P}\underbrace{\left(\frac{B_n}{\sqrt{n}} \geq k\right)}_{:=A_{n,k}} > 0.$$

This is independent of n because $B_n/\sqrt{n} \sim B(1)$ for all n. We then have

$$\mathbb{P}(A_{n,k} \text{ i.o.}) \geq \limsup \mathbb{P}(A_{n,k}) = f(k)$$

So

$$\mathbb{P}\left(\limsup_t \frac{B_t}{\sqrt{t}} \geq k\right) \geq f(k) > 0.$$

Since this is a tail event, it must then have probability 1.

1.3 Stopping times for Brownian motion

Define the σ-field $\mathcal{F}_s$ to be the smallest σ-field containing $\mathcal{F}_s^+$ and all the null sets. Let's now discuss the issue of stopping times. How should we define this?

We have

$$\{S < t\} = \bigcup_n \left\{S \leq t - \frac{1}{n}\right\} \in \mathcal{F}_t^0,$$

$$\{S \leq t\} = \bigcap_n \left\{S < t + \frac{1}{n}\right\} \in \mathcal{F}_t^+.$$

Since these only disagree on null sets, we are okay taking either definition.

Definition 1.1. A **stopping time** is a random variable $T : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ such that $\{T < t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

If $S^{(n)}$ are stopping times for all $n \in \mathbb{Z}$ and $S^{(n)} \searrow S$ a.s., then S is a stopping time. What does a.s. convergence mean in this context?

$$\mathbb{P}(\{\omega : \lim_{n\to\infty} S^{(n)}(\omega) = S(\omega)\}) = 1.$$

Proof. We can split up an event as

$$\{S < t\} = \bigcup_n \underbrace{\{S_n < t\}}_{\mathcal{F}_t}. \qquad \square$$

Remark 1.1. We have a similar result when $S^{(n)} \nearrow S$.

Proposition 1.1. *Let G be an open or closed set in $\mathbb{R}$. Then $S = \inf\{t : B_t \in G\}$ is a stopping time.*

Proof. If G is open,

$$\{S < t\} = \bigcup_{\substack{t' < t \\ t' \in \mathbb{Q}}} \underbrace{\{B_{t'} \in G\}}_{\in \mathcal{F}_t}.$$

Since this is a countable union, $\{S < t\} \in \mathcal{F}_t$.

If G is closed, we define $U_n = \bigcup_{x \in G} B(x, 1/n)$. Then we can define a stopping time S_n based on U_n, and $S_n \nearrow S$. So S is a stopping time. $\square$